Longitudinal instability for a purely inductive wake function

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A purely inductive wake function was the only known case where a solution of the Haissinski equation did not exist beyond a certain threshold. This is due to the ill-defined treatment of the wake function. The solution is proved to exist beyond the threshold, if we define the wake function physically. The threshold of the stability for the solution exists but is lower than the former ''threshold.'' [S1063-651X(99)13808-X]

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Recently, the demand for shorter bunches becomes serious for the better performance of e^+e^- colliding rings, high brightness light sources, the free electron lasers, etc. A better understanding of the bunch instability becomes more important.

It was believed that the Haissinski equation [1] with a purely inductive wake function had no solutions beyond a threshold [2]. However, this was shown to come from an ill-defined treatment of this wake function [3]. By introducing a physical regularization of its singularity, we can prove the existence of the solution. Some authors claim that the threshold of the existence of the Haissinski equation has physical meaning. That is, beyond the threshold, the bunch is unstable. This argument is apparently wrong. The existence of a stationary solution and its stability are not the same. In this paper, we show that the existence of a stationary solution and its stability are not the same, and that the threshold of the existence of the Haissinski equation without regularization has no physical meaning.

This case is of practical importance: wake functions for the modern storage and damping rings tend to be purely inductive [4], because their vacuum chambers tend to be smooth. They contain small discontinuities only, such as shallow steps, transition masks and bellows, etc.

Electrons in an accelerator are enclosed in metals (vacuum pipe, the RF cavities, etc.), and interact with their environment. This effect is represented by the wake field [2]. The wake field acting on an electron is determined by the distribution of electrons ahead of it. At the same time, the distribution of electrons is influenced by the wake field. Hence, to determine the distribution function, one should solve coupled nonlinear equations. The single-particle equations of motion are as follows:

$$\frac{d}{d\bar{s}}\xi=\frac{\alpha}{c}p, \tag{1}$$

$$\frac{d}{d\bar{s}}p=-\frac{\alpha}{c}\xi+\frac{e^2LN\alpha}{\omega_s\sigma_\epsilon T_0E_0c}\int_\xi^\infty d\xi'\rho(\xi')W(\xi'-\xi). \tag{2}$$

Here, the variable $\bar{s}$, and the dimensionless parameters ξ and p are defined as

$$\bar{s}=\frac{\omega_s}{\alpha}s,\quad \xi\equiv\frac{\omega_s}{\alpha\sigma_\epsilon}\tau,\quad p=-\frac{\epsilon}{\sigma_\epsilon}, \tag{3}$$

where c is the velocity of light, e is the electric charge of the electron, L is the total length of the pipe structure in which the wake field is generated, E_0 is the reference energy of the beam, N is the number of electrons in a bunch, T_0 is the revolution period of the beam, τ is the time displacement between an electron and the reference synchronous particle, ϵ is the relative energy $(E-E_0)/E_0$ with E being the electron energy, σ_ϵ is the nominal rms relative energy spread, ω_s is the synchrotron oscillation frequency, α is the momentum compaction factor, and s is the longitudinal coordinate along the ring. The second term of Eq. (2) is the retarding force seen by a particle at ξ due to the longitudinal wake force, which is produced by all the particles in front of it; $\rho(\xi)$ is the particle density at location ξ.

In the presence of radiation, two more parameters are necessary: b is the damping coefficient and $D=b\sigma_\epsilon^2$ is the diffusion coefficient representing the amount of quantum excitation due to photon emission. The dynamics with radiation can be described by the Fokker-Planck equation [5] for the phase-space particle distribution $\psi(\xi,p,\bar{s})$.

$$\frac{\partial\psi}{\partial\bar{s}}=p\frac{\partial}{\partial\xi}\psi+\frac{\alpha b}{\omega_s}\frac{\partial}{\partial p}p\psi-\left(\xi-\frac{e^2LN}{\omega_s\sigma_\epsilon T_0E_0}\times\int_\xi^\infty d\xi'\rho(\xi')W(\xi'-\xi)\right)\frac{\partial}{\partial p}\psi+\frac{\alpha D}{\omega_s\sigma_\epsilon^2}\frac{\partial^2}{\partial p^2}\psi. \tag{4}$$

The static solution is given by

$$\psi_0(\xi,p)=\exp\left(-\frac{p^2}{2}\right)\rho(\xi), \tag{5}$$

$$\rho(\xi)=A\exp\left(-\frac{\xi^2}{2}+\int_\xi^\infty d\xi' V(\xi')\right), \tag{6}$$

$$V(\xi)=\int_\xi^\infty d\xi'\rho(\xi')w(\xi'-\xi), \tag{7}$$

$$w(\xi'-\xi)=-\frac{e^2LN}{\omega_s\sigma_\epsilon T_0E_0}W(\xi'-\xi). \tag{8}$$

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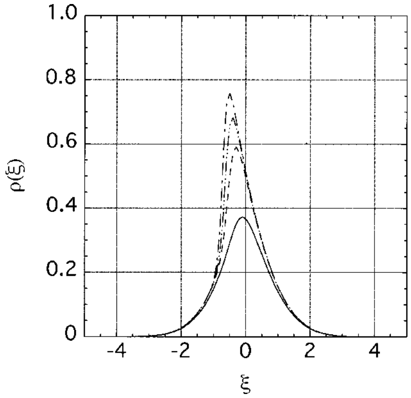

FIG. 1. Solutions of Eq. (13) with the boundary condition Eq. (9) at $a=0.1$ and $S=2$ for $Q=0.7$ (—), $Q=0.85$ (− − −), $Q=0.98$ (− · · · − · · · −), and $Q=1.049$ (− · − · −).

Equation (6) is the Haissinski equation [1] (A is the normalization constant where $\int\rho\, d\xi=1$). Since $\rho(\xi)$ depends only on $\rho(\xi')$ for $\xi<\xi'$, and we know

$$\rho(\xi)\sim A\exp\left(-\frac{\xi^2}{2}\right),\quad \xi\to\infty \tag{9}$$

Eq. (6) can be integrated from the head of the bunch to the tail for a given value A [6]. Let us call the result of such an integration $\rho(\xi;A)$ and define the ''charge'' Q as

$$Q=Q(A)=\int_{-\infty}^{\infty}\rho(\xi;A)d\xi. \tag{10}$$

If a value A exists such that $Q(A)=1$, it gives the solution of the Haissinski equation. Usually, we find the solution of Eq. (10) numerically.

A purely inductive wake function is written as

$$w(\xi)=S\delta'(\xi). \tag{11}$$

This violates the causality as defined in Ref. [2]. It seems reasonable to replace δ' by

$$\delta'(\xi)\to\frac{\delta(\xi)-\delta(\xi-a)}{a}. \tag{12}$$

The δ' wake function is regained when $a\to 0$. Note that a must be positive in order to satisfy the causality condition.

According to Eqs. (6) and (12), we obtain

$$\frac{\rho'}{\rho}=-\xi+\frac{S}{a}[\rho(\xi+a)-\rho(\xi)]. \tag{13}$$

When a is small and S is smaller than its threshold $S_{max}\simeq 1.55$ [2,3], the solution of Eq. (13) can be well approximated by

$$\ln\rho-S\rho=-\tfrac{1}{2}\xi^2+\ln A, \tag{14}$$

where A is determined by a normalization condition. When a is small and S is larger than its threshold $S_{max}\simeq 1.55$, we numerically obtain its solution. In Fig. 1, we show the solution manifold for the case $a=0.1$ and $S=2$. We can numerically confirm that there always exists a solution that satisfies $Q=1$ for arbitrary S and a [3]. Then, we obtain the solution with physical and appropriate regularization of this wake function.

Here, we have to investigate the stability condition. When we consider the case in which the instabilities occur in a time shorter than the damping or diffusion time, the instability condition can be obtained by the Vlasov equation [7]. The Vlasov equation can be obtained by putting $b=D=0$ in Eq. (4). We obtain

$$\frac{\partial}{\partial\bar{s}}\psi(\xi,p,\bar{s})=p\frac{\partial}{\partial\xi}\psi(\xi,p,\bar{s})-[\xi+V(\xi)]\frac{\partial}{\partial p}\psi(\xi,p,\bar{s}). \tag{15}$$

Following the method of Oide and Yokoya [6,8], the stability of $\psi(\xi,p,\bar{s})$ is examined by a linear perturbation. We expand

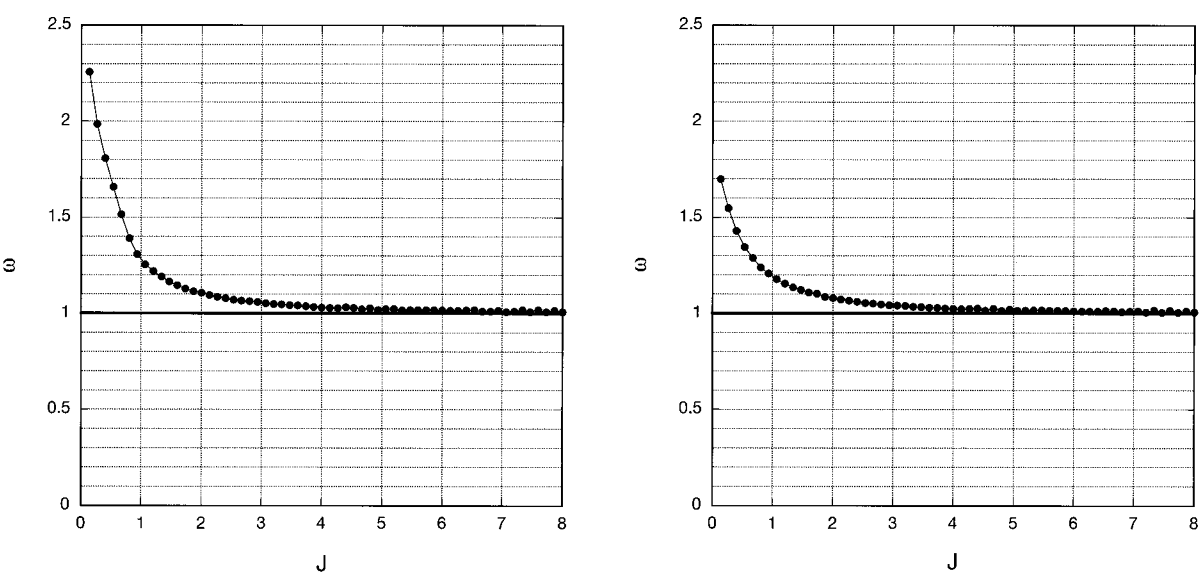

FIG. 2. (Left) Frequency $\omega(J)$ as a function of J at $S=2$ and $a=0.1$ and (right) at $S=1.5$ and $a=0.1$.

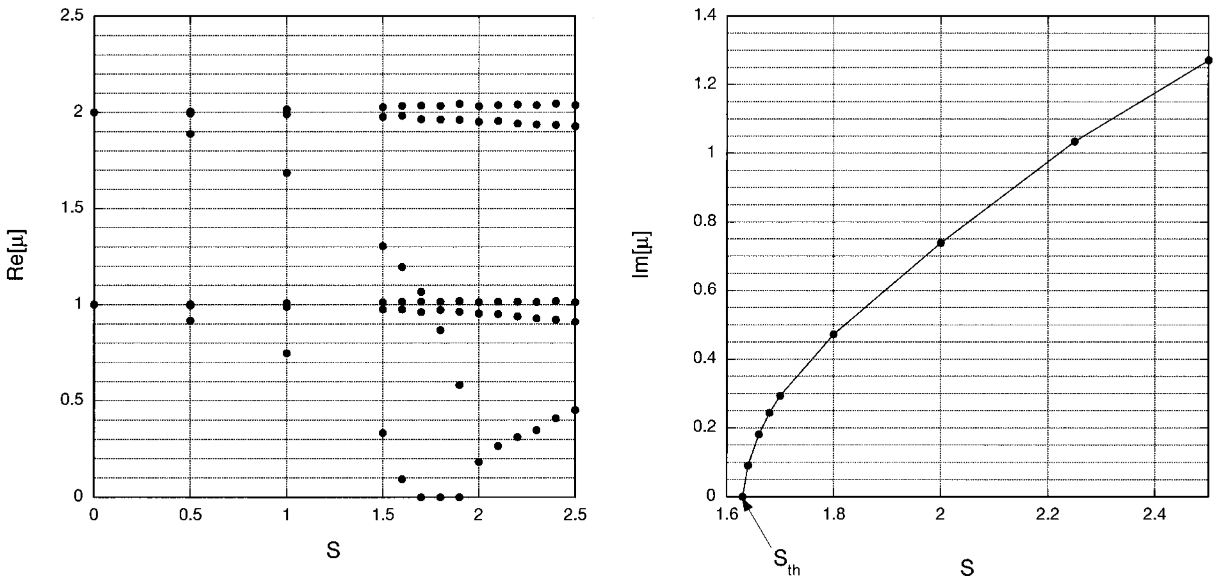

FIG. 3. (Left) Re[μ] spectrum and (right) Im[μ] spectrum at $a=0.1$.

$\psi(\xi,p,\overline{s})$ around its static solution as $\psi(\xi,p,\overline{s})=\psi_0(\xi,p)+\psi_1(\xi,p,\overline{s})$. The first-order terms of $\psi_1(\xi,p,\overline{s})$ can be written as

$$\frac{\partial}{\partial \overline{s}}\psi_1=p\frac{\partial}{\partial \xi}\psi_1-[\xi+V(\xi)]\frac{\partial}{\partial p}\psi_1-V_1(\xi,\overline{s})\frac{\partial \psi_0}{\partial p}, \tag{16}$$

where $V_1(\xi,\overline{s})$ is defined as a wake potential, which is induced by ψ_1.

In order to solve Eq. (16), we introduce the action-angle variables (J,ϕ). Since the Hamiltonian H depends only on J, Eq. (16) is rewritten as follows:

$$\frac{\partial}{\partial \overline{s}}\psi_1=\omega(J)\frac{\partial}{\partial \phi}\psi_1+pV_1(\xi,\overline{s})\psi_0(J), \tag{17}$$

where $\omega(J)=d\phi/d\overline{s}=\partial H/\partial J$ and $\partial\psi_0/\partial p=-p\psi_0$. We select the origin of ϕ as the ξ axis. We now expand ψ_1 by the orthogonal mode as follows:

$$\psi_1=\sum_{nm} m\omega(J_n)i\sqrt{\psi_0\Delta J_n}(C_{nm}\cos m\phi+S_{nm}\sin m\phi)\Delta_n(J)\exp(-i\mu\overline{s}), \tag{18}$$

where we introduced the function $\Delta_n(J)$, which has the value $1/\Delta J_n$ in the strip around the nth mesh point $J=J_n$ with thickness ΔJ_n, and zero outside.

Substituting Eq. (18) into Eq. (17), we obtain the matrix equation

$$-\mu^2 C_{nm}=-\sum_{n'm'} M_{nmn'm'}C_{n'm'} \tag{19}$$

with

$$M_{nmn'm'}=m^2\omega_n^2\delta_{nn'}\delta_{mm'}-\frac{mm'\omega_n\omega_{n'}\sqrt{\psi_0(J_n)\Delta J_n}\sqrt{\psi_0(J_{n'})\Delta J_{n'}}}{\pi}\times\int_0^{2\pi}\int_0^{2\pi}\cos m\phi\cos m'\phi' F(q(J_n',\phi')-q(J_n,\phi))d\phi\, d\phi', \tag{20}$$

where $\delta_{nn'}$ is the Kronecker delta and F is a primitive function that satisfies $F'(\xi)=w(\xi)$. When eigenvalues of this matrix are negative or complex, the system becomes unstable. It is generally difficult to study the stability condition for the general wake function in an analytical way. It is usually studied numerically [6,8,9].

In order to calculate the matrix M for a purely inductive wake function in Eq. (20), we select $S/a\times[\theta(\xi)-\theta(\xi-a)]$ as a primitive function F. Further, we need to know

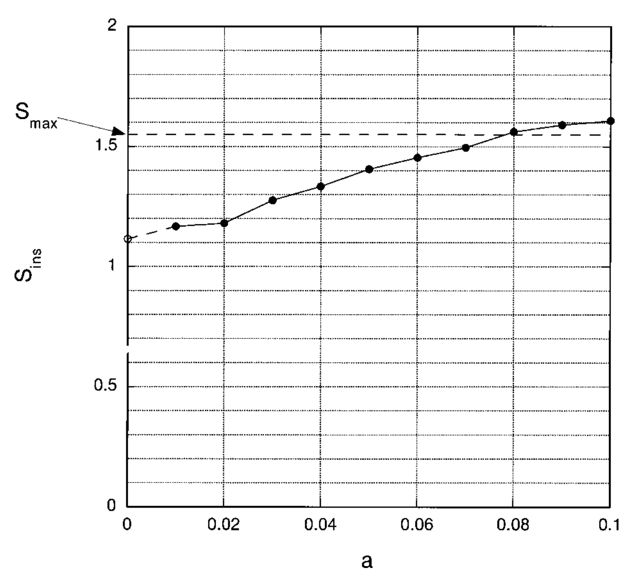

FIG. 4. S_{ins}-a relation.

how the Hamiltonian H depends on the action variable J. In this system the Hamiltonian $H(\xi,p)$ for the single-particle motion in the bunch is described as follows:

$$H(\xi,p)=\frac{p^2}{2}+\frac{\xi^2}{2}-S\int_{\xi}^{\infty}d\xi'\frac{\rho(\xi')-\rho(\xi'+a)}{a}. \quad (21)$$

The potential term is obtained numerically by integrating the static solution. According to the definition of the action variable J,

$$J=\frac{2}{2\pi}\int\sqrt{2\left(H-\frac{\xi^2}{2}+S\int_{\xi}^{\infty}d\xi'\frac{\rho(\xi')-\rho(\xi'+a)}{a}\right)}d\xi, \quad (22)$$

which represents the relation between J and H. The frequency $\omega(J)$ is obtained by $\partial H/\partial J$. The behavior of $\omega(J)$ is shown in Fig. 2.

Figure 3 represents the spectra of the eigenvalues μ (the square root of the eigenvalues of M). The threshold ''S_{ins}'' is calculated by square-root fitting. In this case the instability appears around $S \geqslant S_{ins} \simeq 1.60872$ with $a=0.1$. Figure 4 represents how S_{ins} depends on a. This suggests that the threshold S_{max} is different from S_{ins}.

This numerical analysis suggests that with physical and appropriate regularization of this wake function, the solution exists. Further, a numerical analysis shows that there is (S_{ins}) beyond which the solution becomes unstable. This value of (S_{ins}) is lower than the unphysical threshold (S_{max}). This means that the nonexistence of the solution for the Haissinski equation with the nonregularized purely inductive wake function has no physical meaning.

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